Impossibility of calculating magnetic field change from current disruption

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Abstract: The picture of the substorm current wedge, formed by visualizing the cross-tail current as reduced or disrupted and thus diverted through the ionosphere, provides a compact summary of the magnetic field changes observed during substorms. There has long been a tendency, however, to view current disruption as an actual explanation, not just a convenient representation, of the magnetic field changes — to search for some model by which first to predict the current disruption and then, as a consequence, to calculate the magnetic field dipolarization from the Biot-Savart integral over the reduced current. Formally, the time derivative of the magnetic field can be expressed as the Biot-Savart integral over the time derivative of the current density, which in turn can be calculated in principle by summing all the forces (weighted by charge/mass) on all the charged particles. In the resulting expression, the integrand includes an electric field term which can be transformed (by means of an integration by parts) into curl $\mathbf{E}$. Thus, the time derivative of $\mathbf{B}$ cannot be calculated directly from the Biot-Savart integral because one term in the integrand contains the time derivative itself, and the contribution of that term is very large when the electron inertial length is small in comparison to the spatial scale of the system; instead, the time derivative of $\mathbf{B}$ must be calculated by solving what is now an integral equation. In the limit of small electron inertial length, the solution reduces to the curl of all the terms other than $\mathbf{E}$; this is identical to the method described by Vasyliūnas [9, 10] for obtaining the time evolution of $\mathbf{B}$ — determined directly by plasma dynamics through the generalized Ohm's law and not by the changing current (which cannot be calculated except as the time derivative of curl $\mathbf{B}$).

Key words: current disruption, dipolarization, magnetic field change, substorm expansion.

1. Introduction

The notion that magnetic fields and their changes are to be understood by reference to electric currents is deeply ingrained in the thinking of many researchers on the magnetosphere. In particular, the striking phenomenon known as dipolarization of the magnetic field in the nightside magnetosphere, observed in association with the substorm expansion, is widely interpreted as the formation and evolution of an (inferred) substorm current wedge (e.g. [5]): the cross-tail current is reduced over a limited local time sector by having part of the current flow down along magnetic field lines to the ionosphere, westward across the ionosphere, and back up along the field lines. The process is often referred to as an example of "current disruption," and much of the modeling under that label would seem to be aimed at predicting the formation and subsequent evolution of the current wedge, from which the dipolarization of the magnetic field could then be deduced.

A basic presumption of such an approach is that Ampère's law

$$\mathbf{J} = \frac{c}{4\pi}\,\nabla \times \mathbf{B} \tag{1}$$

(I use Gaussian units throughout) determines the magnetic field $\mathbf{B}$ given the current density $\mathbf{J}$, with the further implicit understanding that this holds for time variations as well: to determine the time evolution of $\mathbf{B}$, one seeks first to specify the time evolution of $\mathbf{J}$. The contrary view, that Ampère's law determines $\mathbf{J}$ given $\nabla \times \mathbf{B}$, has long been a familiar concept within magnetohydrodynamics [1, 2, 6, 8], where the time evolution of $\mathbf{B}$ is taken as determined by Faraday's law

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E} \tag{2}$$

with $\mathbf{E}$ given, in the simplest case, by the MHD (frozen-flux) approximation in terms of the plasma bulk flow. The undeniable importance of non-MHD effects in some aspects of the substorm process, however, has been invoked as an argument for ignoring any MHD constraints.

In two recent papers [9, 10], I have examined the time evolution and interrelationships of $\mathbf{E}$, $\mathbf{J}$, and $\mathbf{B}$ on the basis of the exact fundamental equations and have shown that, provided $\mathbf{E}$ is calculated from the full generalized Ohm's law rather than just the MHD approximation, the time evolution of $\mathbf{B}$ is determined by Faraday's law (2), not by the time derivative of Ampère's law (1) (which serves instead to determine the time evolution of $\mathbf{J}$ from that of $\mathbf{B}$), and that this (nominally large-scale) approach remains valid on space and time scales down to those of electron plasma oscillations, thus extending well beyond the range of MHD (generally considered no longer applicable once scales as small as ion gyroperiod or ion inertial length are approached); it is limited ultimately by the breakdown of charge quasineutrality, not of the frozen-flux approximation. Concerning current disruption, I summarized the conclusion of [9] as follows: "The results in the present paper imply that any such theoretical model of dipolarization, in terms of the current as the primary quantity, is not possible: *on time scales appropriate to substorm expansion, there is no equation from which the time evolution of the current could be calculated, prior to and independently of* $\nabla \times \mathbf{B}$. ...These limitations apply to any attempts at accounting for changing magnetic fields by invoking changing currents — current disruption, diversion, wedge

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formation, etc. Over the wide range of time scales from electron plasma period to Alfvén wave travel time, there simply is no way to calculate the changing currents except by taking the curl of the changing magnetic fields; statements about changes of current are not explanations but merely descriptions of changes in the magnetic field."

2. Evolution of electric current

Note that the above conclusion is a very specific one: within the stated range of length and time scales, there is no usable equation from which one could calculate the time evolution of the current independently, i.e., other than from $(\partial/\partial t)\nabla\times\mathbf{B}$. It is thus absolutely pointless, for anyone who wants to question the conclusion, to talk about approaches or paradigms and to invoke general arguments such as those in the controversy [6, 7, 8, 3, 4] on whether the magnetic field and the plasma flow or the electric current and the electric field are to be treated as the primary variables; rather, the only effective counterargument is to write down what one claims to be the usable independent equation for $\partial\mathbf{J}/\partial t$.

An independent equation for $\partial\mathbf{J}/\partial t$ always exists, of course, in principle: with the current density obtained by summing the motions of all the charged particles, its time rate of change can be determined by summing the accelerations of all the charged particles. In terms of velocity distribution functions, $\mathbf{J}$ is defined by

$$\mathbf{J} = \sum_a q_a \int d^3v\, \mathbf{v} f_a(\mathbf{v}) \tag{3}$$

where $f_a(\mathbf{v})$ is the velocity distribution function of charged particles of species a. The equation for time evolution of $\mathbf{J}$ can then be calculated from the appropriate sum of velocity-moment equations (see, e.g., [9] and references therein)

$$\frac{\partial\mathbf{J}}{\partial t} = \sum_a \left\{ \frac{q_a^2 n_a}{m_a}\left(\mathbf{E} + \frac{\mathbf{V_a}}{c}\times\mathbf{B}\right) - \frac{q_a}{m_a}\left(\nabla\cdot\kappa_a\right) + q_a n_a \mathbf{g} \right\} + \left(\frac{\delta\mathbf{J}}{\delta t}\right)_{coll} \tag{4}$$

where q_a, m_a, n_a, $\mathbf{V_a}$, and κ_a are the charge, mass, concentration, bulk velocity, and kinetic tensor, respectively, of species a, $\mathbf{g}$ is the gravitational acceleration (included here for exactness but, as far as phenomena in the terrestrial magnetosphere are concerned, mostly not important in practice), and $(\delta\mathbf{J}/\delta t)_{coll}$ represents the sum of all collision effects. Except for being non-relativistic, equation (4) is exact, with no approximations.

The essential point demonstrated in [9] is that while equation (4) always holds *in principle*, its left-hand side becomes negligibly small in comparison to the individual terms on the right-hand side, except when variations on space and time scales at and below those of electron plasma oscillations are involved; on all larger scales the equation is thus *in practice* not usable for determining $\partial\mathbf{J}/\partial t$. When small-scale variations are important, they can be averaged over, and equation (4) can be transformed into the corresponding equation for the time evolution of the averaged $\mathbf{J}$ (expressed in terms of average quantities and fluctuation correlations):

$$\begin{aligned}\frac{\partial\langle\mathbf{J}\rangle}{\partial t} = \sum_a \Bigg\{ & \frac{q_a^2\langle n_a\rangle}{m_a}\left(\langle\mathbf{E}\rangle + \frac{\langle\mathbf{V_a}\rangle}{c}\times\langle\mathbf{B}\rangle\right) \\ &+ \frac{q_a^2}{m_a}\left(\langle\delta n_a \delta\mathbf{E}\rangle + \langle\frac{\delta\left(n_a\mathbf{V_a}\right)}{c}\times\delta\mathbf{B}\rangle\right) \\ &- \frac{q_a}{m_a}\nabla\cdot\langle\kappa_a\rangle + q_a\langle n_a\rangle\mathbf{g}\Bigg\} + \langle\left(\frac{\delta\mathbf{J}}{\delta t}\right)_{coll}\rangle .\end{aligned} \tag{5}$$

Equation (5) is still exact (except for being non-relativistic) and in particular does *not* presuppose any small-amplitude or quasilinear approximation (as long as the average moments are properly defined as moments of the averaged distribution function [9]).

It is convenient to rewrite (5) in a simplified notation as

$$\frac{\partial\langle\mathbf{J}\rangle}{\partial t} = \frac{{\omega_p}^2}{4\pi}\left(\langle\mathbf{E}\rangle - \langle\mathbf{E}^*\rangle\right) \tag{6}$$

where the effective electron plasma frequency ω_p is defined by

$${\omega_p}^2 \equiv 4\pi\sum_a \frac{q_a^2\langle n_a\rangle}{m_a} \approx \frac{4\pi n_e e^2}{m_e} \tag{7}$$

and $-\langle\mathbf{E}^*\rangle$ represents the sum of all the terms on the right-hand side other than $\langle\mathbf{E}\rangle$; this is purely a matter of notation and does not presuppose any restrictions.

3. Evolution of magnetic field

The following argument can be (and has been) made: regardless of any conclusions in [9] about orders of magnitude and small-scale fluctuations, equations (4) and (5) do represent, formally at least, the time evolution of the current density, so why can they not be used to calculate the time evolution of the magnetic field? Equation (4) includes all space and time scales (including those that may be considered too small to be of interest) and describes, strictly speaking, every individual plasma oscillation, but if that is perceived as a problem, then the averaged equation (5) can always be used instead. I consider now the consequences of applying this seemingly straightforward procedure.

3.1. Application of Biot-Savart law

Solved for $\mathbf{B}$ in terms of $\mathbf{J}$, Ampère's law (1) yields the Biot-Savart integral

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\int d^3r'\, \mathbf{J}(\mathbf{r}',t)\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{8}$$

which, by a simple integration by parts, can be rewritten in the form

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\int d^3r'\, \frac{\nabla'\times\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} \tag{9}$$

($\nabla' =$ gradient with respect to the coordinate vector $\mathbf{r}'$). Differentiating with respect to time gives

$$\frac{\partial\mathbf{B}(\mathbf{r},t)}{\partial t} = \frac{1}{c}\int d^3r'\, \frac{\partial\mathbf{J}(\mathbf{r}',t)}{\partial t}\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{10}$$

or equivalently

$$\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} = \frac{1}{c}\int d^3r' \, \frac{\nabla' \times \partial \mathbf{J}(\mathbf{r}',t)/\partial t}{|\mathbf{r}-\mathbf{r}'|} \tag{11}$$

and the idea is to calculate $\partial \mathbf{B}/\partial t$ by using (5) for $\partial \mathbf{J}/\partial t$ within the integrals. (It is taken for granted that the time variations of interest here occur on scales long compared to light travel times; hence the neglect of the displacement current term in Ampère's law and consequently of time retardation in the integrals.)

Substituting $\partial \mathbf{J}/\partial t$ from (5) and invoking Faraday's law (2) to evaluate $\nabla \times \mathbf{E}$ transforms the Biot-Savart integral (11) for $\partial \mathbf{B}/\partial t$ into

$$\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} = -\int d^3r' \, \frac{\partial \mathbf{B}(\mathbf{r}',t)/\partial t + \nabla' \times c\mathbf{E}^*(\mathbf{r}',t) - \Delta}{4\pi{\lambda_e}^2|\mathbf{r}-\mathbf{r}'|}$$

$$\Delta \equiv (\nabla' n_e/n_e) \times c(\mathbf{E}-\mathbf{E}^*) \tag{12}$$

where

$$\lambda_e \equiv c/\omega_p = 5 \text{ km } (1 \text{ cm}^{-3}/n_e)^{1/2} \tag{13}$$

is the electron inertial length (also known as the collisionless skin depth). The difficulty is now apparent: $\partial \mathbf{B}/\partial t$ cannot be calculated simply by evaluating the integral in (12) because the integrand contains $\partial \mathbf{B}/\partial t$ itself as one of the terms. Nor can this term be considered as a small correction: the order of magnitude of the integral over $\partial \mathbf{B}/\partial t$ on the right-hand side, compared to the term $\partial \mathbf{B}/\partial t$ on the left-hand side, is $O(\mathcal{L}/\lambda_e)^2$, where $\mathcal{L}$ is the spatial scale of the system. Equation (12) must in fact be viewed as an integral equation for $\partial \mathbf{B}/\partial t$, not just a plain integral.

3.2. Large-scale limit

The integral equation (12) can be solved explicitly for $\partial \mathbf{B}/\partial t$ if λ_e varies only on a spatial scale large compared to itself ($\lambda_e \ll \mathcal{L}$); to lowest order in $\lambda_e/\mathcal{L}$,

$$\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} = -\int d^3r' \exp\left\{\frac{-|\mathbf{r}-\mathbf{r}'|}{\lambda_e}\right\} \frac{\nabla' \times c\mathbf{E}^*(\mathbf{r}',t)}{4\pi{\lambda_e}^2|\mathbf{r}-\mathbf{r}'|} \tag{14}$$

(the term Δ in (12) has been neglected, as it can be shown to be of order $(\lambda_e/\mathcal{L})^2$ in comparison to the others). The solution (14) is most readily derived by transforming the integral equation (12) back into a differential form by making use of the fact that the Green's function $\psi = 1/|\mathbf{r}-\mathbf{r}'|$ is a solution of

$$\nabla^2\psi = -4\pi\,\delta\,(\mathbf{r}-\mathbf{r}') \tag{15}$$

and then placing all the $\partial \mathbf{B}/\partial t$ terms in the consequent differential equation on its left-hand side, with the result that the Green's function is now a solution of

$$\nabla^2\psi - \frac{\psi}{{\lambda_e}^2} = -4\pi\,\delta\,(\mathbf{r}-\mathbf{r}') \tag{16}$$

instead of (15); if λ_e can be treated (locally at least) as a constant, the solution of (16) is the well-known Debye or Yukawa potential. Alternatively, the differential form of the equation for $\partial \mathbf{B}/\partial t$ can be obtained directly from the curl of the time derivative of Ampère's law (1), with the use of (5) and (2).

Equation (14) expresses $\partial \mathbf{B}/\partial t$ as a straightforward integral (one that no longer contains $\partial \mathbf{B}/\partial t$ itself in the integrand). It differs from (12) also in the form of the kernel (Green's function): the Coulomb potential in (12) has been replaced in (14) by a potential of the Debye form (but note that the shielding distance here is the electron inertial length λ_e, *not* the Debye length).

Changing the variable of integration from $\mathbf{r}'$ to $\mathbf{s}$ with $\mathbf{r}' \equiv \mathbf{r} + \lambda_e\mathbf{s}$ and writing the integral over $\mathbf{s}$ in spherical coordinates finally gives

$$\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} = -\int \frac{d\Omega}{4\pi}\int_0^\infty s\,ds\,e^{-s}\nabla \times c\mathbf{E}^*(\mathbf{r}+\lambda_e\mathbf{s},t). \tag{17}$$

In the limit $\lambda_e \ll \mathcal{L}$ this reduces to

$$\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} \approx -\nabla \times c\mathbf{E}^*(\mathbf{r},t) \tag{18}$$

which is equivalent to

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E} \quad \text{with} \quad 0 = \mathbf{E} - \mathbf{E}^* \tag{19}$$

But this is precisely the method of calculating the time evolution of $\mathbf{B}$ arrived at in [9, 10]: on length scales $\gg \lambda_e$ and time scales $\gg 1/\omega_p$, the electric field is determined by plasma dynamics via the generalized Ohm's law (neglecting the $\partial \mathbf{J}/\partial t$ term), and the evolution of the magnetic field is then determined, via Faraday's law, directly by the curl of the electric field. There is no longer any direct reference to the electric current density, which is determined — and this is now the only role of Ampère's law — by the curl of the magnetic field.

4. Conclusion

The presence of a large concentration of free charged particles (particularly electrons) in a plasma means that an electric field can, by accelerating positive and negative charges in opposite directions, very quickly and efficiently change the electric current density — unless other forces (e.g. magnetic forces or pressure gradients) counteract this differential acceleration. What constitutes a "large" concentration in this context is defined precisely by the value of n_e implied by the condition $\lambda_e \ll \mathcal{L}$: when this condition is satisfied, a very large current density can result from even a small differential acceleration of positive and negative particles, with the result that the electric field must be determined largely by the requirement that the differential acceleration remain sufficiently close to zero. This is the basic reason why the time evolution of the current cannot be specified independently of and logically prior to the time evolution of the magnetic field: if the change of current is assumed to be specified somehow, then Ampère's law implies a change of the magnetic field, which by Faraday's law must be accompanied by a (non-curl-free) electric field, which implies in turn a change of current, much larger than (and hence inconsistent with) that assumed initially.

Here I have demonstrated this inconsistency by an explicit calculation: in order to obtain the time derivative of the magnetic field, insert the changing current density, deduced from the forces acting on all the charged particles, into the time derivative of the Biot-Savart integral. Depending on how one handles the mathematics, there are two possible results. Either, if the integral is simply evaluated as given, one finds that the time evolution of the magnetic field cannot be calculated at all unless it is known already (and known indeed to a much higher precision, $\ll O(\lambda_e/\mathcal{L})^2$, than that of the result to be calculated). Or else, if the appropriate mathematical manipulations are carried out, one *can* obtain the time evolution of the magnetic field, but (one finds) it is actually being calculated from the changes in the balance (described by the generalized Ohm's law) between the electric field and the plasma flows and stresses: even though the Biot-Savart integral was taken as the starting point, the final formula arrived at for the time derivative of the magnetic field gives it directly as minus the curl of the electric field derived from the generalized Ohm's law, *not* as the integrated magnetic effect of any specified varying currents.

References

1. Cowling, T. G., *Magnetohydrodynamics*, Section 1.3, Interscience Publishers, Inc., New York, 1957.
2. Dungey, J. W., *Cosmic Electrodynamics*, p.10, Cambridge University Press, London, 1958.
3. Heikkila, W. J., Comment on "The alternative paradigm for magnetospheric physics" by E. N. Parker, *J. Geophys. Res., 102*, 9651–9656, 1997.
4. Lui, A. T. Y., Electric current approach to magnetospheric physics and the distinction between current disruption and magnetic reconnection, in *Magnetospheric Current Systems*, edited by S.-I. Ohtani, R. Fujii, M. Hesse, and R. L. Lysak, pp. 31–40, AGU Geophysical Monograph 118, Washington, D.C., 2000.
5. McPherron, R. L., Russell, C. T., and Aubry, M. P., Satellite studies of magnetospheric substorms on August 15, 1968, 9. Phenomenological model for substorms, *J. Geophys. Res., 78*, 3131–3149, 1973.
6. Parker, E. N., The alternative paradigm for magnetospheric physics, *J. Geophys. Res., 101*, 10587–10625, 1996.
7. Parker, E. N., Reply, *J. Geophys. Res., 102*, 9657–9658, 1997.
8. Parker, E. N., Newton, Maxwell, and magnetospheric physics, in *Magnetospheric Current Systems*, edited by S.-I. Ohtani, R. Fujii, M. Hesse, and R. L. Lysak, pp. 1–10, AGU Geophysical Monograph 118, Washington, D.C., 2000.
9. Vasyliūnas, V. M., Time evolution of electric fields and currents and the generalized Ohm's law, *Ann. Geophys., 23*, 1347–1354, 2005a.
10. Vasyliūnas, V. M., Relation between magnetic fields and electric currents in plasmas, *Ann. Geophys., 23*, 2589–2597, 2005b.